## Today's AGENDA

## - FRAPPY Time!

- Warm-UP problems
- Chapter 4: Normal Distributions
- REWARD/ Review Day is Friday, Sept 27 or Mon. Sept 30- BE prepared if you plan to retake the test


# Chapter 4: Numerical Methods for Section 4.4 Distributions of Data 

Interpreting Center \& Variability in a Distribution
Statistics and Data Analysis, $5^{\text {th }}$ edition - For AP* PECK, OLSEN, \& DEVORE

# Chapter 4-Section 4.4-4.5 Modeling Distributions of Data 

Learning Objectives: I can...

- Describe Location in a Distribution
- define and use Density Curves
understand Normal Distributions
use the Empirical Rule
Calculate percentages using z Scores
understand Chebyshev's Rule


# Section 4.4 plus Density Curves Describing Location in a Distribution 

## Learning Objectives: I can...

After this section, you should be able to...
$\checkmark$ MEASURE position using percentiles
$\checkmark$ MEASURE position using $z$-scores
$\checkmark$ TRANSFORM data
$\checkmark$ DEFINE and DESCRIBE density curves

## Warm-UP

What is 5 meters measured in feet?
Define the standard deviation of a sample.
3. How does variance relate to standard deviation?
4. What is random? What is normal?

## Narnoup

1) What is 5 meters measured in feet?

Conversion factor: $1=\frac{1 \text { meter }}{3.28 \mathrm{ft}}$ or $1=\frac{3.28 \mathrm{ft}}{1 \text { meter }}$

$$
\text { so } 5 \mathrm{~m} \times \frac{3.28 \mathrm{ft}}{1 \text { meter }} \approx 16.4 \mathrm{ft}
$$

2) Define the standard deviation of a sample:

$$
s_{x}=\text { sample standard deviation }
$$

A statistic that measures the typical distance from the mean for values (observations) in a distribution. It is calculated by finding the "average" of the squared distances, and then taking the square root

## Warm- UP (cont.)

3) How does variance relate to standard deviation?

Variance is the squared value of standard deviation, or
standard deviation is the square root of variance
(s.d. $)^{2}=$ variance or $\sqrt{\text { variance }}=s . d$.

## Notation for Variance and Standard Deviation: Sample vs. Population

## SAMPLE Notation

- Sample mean: $\bar{x}$
-Sample Variance:
$s_{x}^{2}$ or $\boldsymbol{s}^{2}=\frac{\sum\left(x_{i}-\bar{x}\right)^{2}}{n-1}$
Sample S.D.
$s_{x}$ or $s=\sqrt{\frac{\sum\left(x_{i}-\bar{x}\right)^{2}}{n-1}}$

Population Notation

- Population Mean: $\mu$
- Population Variance:

$$
\sigma^{2}=\frac{\sum(x-\mu)^{2}}{N}
$$

- Population S.D.:

$$
\sigma=\sqrt{\frac{\sum(x-\mu)^{2}}{N}}
$$

## Measuring Position: Percentiles

One way to describe the location of a value in a distribution is to tell what percent of observations are less than it.

## Definition:

The $p^{\text {th }}$ percentile of a distribution is the value with $p$ percent of the observations less than it.

## Example

Jenny earned a score of 86 on her test. How did she perform relative to the rest of the class?

| 6 | 7 |
| :--- | :--- |
| 7 | 2334 |
| 7 | 5777899 |
| 8 | 00123334 |
| 8 | 569 |
| 9 | 03 |

Her score was greater than 21 of the 25 observations. Since 21 of the 25 , or $84 \%$, of the scores are below hers, Jenny is at the $84^{\text {th }}$ percentile in the class's test score distribution.

## Mental Math

## Solve these equations for $z$

a) $90=5 z+80$
$z=2$
b) $75=5 z+80$
$z=-1$
c) $95=5 z+80$
$z=3$
d) $68=5 z+80$
$z=-2.4$

# Solve these equations in terms of $z$ 

a) $90=5 z+80 \quad$ a) $z=\frac{90-80}{5}$
b) $75=5 z+80 \quad$ b) $Z=\frac{75-80}{5}$

$$
\begin{array}{ll}
\text { c) } 95=5 z+80 & \text { c) } z=\frac{95-80}{5}
\end{array}
$$

$$
z \text { score }=\frac{\text { observ-mean }}{\text { s.d. }}
$$

## Measuring Position: Z-Scores

A $z$-score tells us how many standard deviations from the mean an observation falls, and in what direction.

## Definition:

If $x$ is an observation from a distribution that has known mean and standard deviation, the standardized value of $x$ is:

$$
z=\frac{x-\text { mean }}{\text { standard deviation }}
$$

A standardized value is often called a $\boldsymbol{z}$-score.

Jenny earned a score of 86 on her test. The class mean is 80 and the standard deviation is 6.07 . What is her standardized score?

$$
z=\frac{x-\text { mean }}{\text { standard deviation }}=\frac{86-80}{6.07}=0.99
$$

## Using $\mathbf{z}$-scores for Comparison

We can use $z$-scores to compare the position of individuals in different distributions.

## Example

Jenny earned a score of 86 on her statistics test. The class mean was 80 and the standard deviation was 6.07 . She earned a score of 82 on her chemistry test. The chemistry scores had a fairly symmetric distribution with a mean 76 and standard deviation of 4 . On which test did Jenny perform better relative to the rest of her class?

## Transforming Data

Transforming converts the original observations from the original units of measurements to another scale. Transformations can affect the shape, center, and spread of a distribution.

## Effect of Adding (or Subtracting) a Constant (a)

Adding the same number a (either positive, zero, or negative) to each observation:
-adds a to measures of center and location (mean, median, quartiles, percentiles), but
-Does not change the shape of the distribution or measures of spread (range, IQR, standard deviation).


## Transforming Data

## Effect of Multiplying (or Dividing) by a Constant

Multiplying (or dividing) each observation by the same number $b$ (positive, negative, or zero):

- multiplies (or divides) measures of center and location by $b$
- multiplies (or divides) measures of spread by |b/, but
- does not change the shape of the distribution



## Transforming Data

## Effect of Multiplying (or Dividing) by a Constant

Multiplying (or dividing) each observation by the same number $b$ (positive, negative, or zero):

- multiplies (or divides) measures of center and location by $b$
- multiplies (or divides) measures of spread by |b/, but
- does not change the shape of the distribution



## Density Curves

- In Chapters $1 \& 3$, we developed a kit of graphical and numerical tools for describing distributions. Now, we'll add one more step to the strategy.


## Exploring Quantitative Data

1. Always plot your data: make a graph.
2. Look for the overall pattern (shape, center, and spread) and for striking departures such as outliers.
3. Calculate a numerical summary to briefly describe center and spread.
4. Sometimes the overall pattern of a large number of observations is so regular that we can describe it by a smooth curve.

## Sample Density Curves

Histogram of Years with density curve




## Density Curve

## Definition:

A density curve is a curve that

- is always on or above the horizontal axis, and
- has area of exactly 1 underneath it.

A density curve describes the overall pattern of a distribution. The area under the curve and above any interval of values on the horizontal axis is the proportion of all observations that fall in that interval.


## Describing Density Curves

- Our measures of center and spread apply to density curves as well as to actual sets of observations.


## Distinguishing the Median and Mean of a Density Curve

The median of a density curve is the equal-areas point, the point that divides the area under the curve in half.
The mean of a density curve is the balance point, at which the curve would balance if made of solid material.
The median and the mean are the same for a symmetric density curve. They both lie at the center of the curve. The mean of a skewed curve is pulled away from the median in the direction of the long tail.


(b)

## Normal Distributions

- One particularly important class of density curves are the Normal curves, which describe Normal distributions.
- All Normal curves are symmetric, single-peaked, and bell-shaped
- Any Specific Normal curve is described by giving its mean $\mu$ ("mu") and standard deviation $\sigma$ ("sigma").


Two Normal curves, showing the mean $\mu$ and standard deviation $\sigma$.

## Definition:

A Normal distribution is described by a Normal density curve. Any particular Normal distribution is completely specified by two numbers: its mean $\mu$ (" $m u$ ") and standard deviation $\sigma$ ("sigma").
-The mean $(\mu)$ of a Normal distribution is the center of the symmetric Normal curve.
-The standard deviation ( $\sigma$ ) is the distance from the center to the change-of-curvature points (points of inflection) on either side.
-We abbreviate the Normal distribution with mean and standard deviation as: $\boldsymbol{N}(\mu, \sigma)$.

Normal distributions are good descriptions for some distributions of real data.
Normal distributions are good approximations of the results of many kinds of chance outcomes.

Most of our statistical inference procedures are based on Normal distributions.

## The Empirical Rule (68-95-99.7 Rule)

Although there are many Normal curves, they all have properties in common.

## Definition: The 68-95-99.7 Rule ("The Empirical Rule")

In the Normal distribution with mean $\mu$ and standard deviation $\sigma$ :
-Approximately $68 \%$ of the observations fall within $1 \sigma$ of $\mu$.
-Approximately $95 \%$ of the observations fall within $2 \sigma$ of $\mu$.
-Approximately $99.7 \%$ of the observations fall within $3 \sigma$ of $\mu$.


## Describing Density Curves



## Review of Learning Objectives Describing Location in a Distribution

## Summary

In this section, we learned that...
$\checkmark$ There are two ways of describing an individual's location within a distribution - the percentile and $\boldsymbol{z}$-score..
$\checkmark$ It is common to transform data, especially when changing units of measurement. Transforming data can affect the shape, center, and spread of a distribution.
$\checkmark$ We can sometimes describe the overall pattern of a distribution by a density curve (an idealized description of a distribution that smooths out the irregularities in the actual data).
$\checkmark$ the Empirical Rule (68-95-99.7 Rule) describes percentage of observations found within intervals for all Normal distributions

## Looking Ahead...

## In the next Section...

We'll learn about one particularly important class of density curves - the Normal Distributions

We'll learn
$\checkmark$ To USE the 68-95-99.7 Rule
$\checkmark$ The Standard Normal Distribution
$\checkmark$ Normal Distribution Calculations, and
$\checkmark$ Assessing Normality

## Section 4.5 Normal Distributions

## Learning Objectives

After this section, you should be able to...
$\checkmark$ DESCRIBE and APPLY the Empirical Rule (68-95-99.7 Rule)
$\checkmark$ DESCRIBE the standard Normal Distribution
$\checkmark$ PERFORM Normal distribution calculations
$\checkmark$ ASSESS Normality

## Normal Distributions

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## Example

The distribution of lowa Test of Basic Skills (ITBS) vocabulary scores for $7^{\text {th }}$ grade students in Gary, Indiana, is close to Normal. Suppose the distribution is $N(6.84,1.55)$. $N(\mu, \sigma)$
Sketch the Normal density curve for this distribution, showing the values at $\pm 1 \sigma, \pm 2 \sigma$, and $\pm 3 \sigma$.


ITBS score

$$
\begin{gathered}
\pm 1 \sigma \rightarrow \mu-1 \sigma \\
\rightarrow \mu+1 \sigma, \text { so } \\
6.84-1.55=5.29 \\
6.84+1.55=8.39 \\
\pm 2 \sigma \rightarrow \mu-2 \sigma \\
\rightarrow \mu+2 \sigma, \text { so } \\
6.84-3.10=3.74 \\
6.84+3.10=9.94 \\
\pm 3 \sigma \rightarrow \mu-3 \sigma \\
\rightarrow \mu+3 \sigma, s o \\
6.84-4.65=2.19 \\
6.84+4.65=11.49
\end{gathered}
$$

## Example

The distribution of lowa Test of Basic Skills (ITBS) vocabulary scores for $7^{\text {th }}$ grade students in Gary, Indiana, is close to Normal. Suppose the distribution is $N(6.84,1.55)$. $N(\mu, \sigma)$
a) Sketch the Normal density curve for this distribution.
b) What percent of ITBS vocabulary scores are less than 3.74 ?
c) What percent of the scores are between 5.29 and 9.94 ?




## The Standard Normal Distribution

All Normal distributions are the same if we measure in units of size $\sigma$ from the mean $\mu$ as center.

## Definition:

The standard Normal distribution is the Normal distribution with mean 0 and standard deviation 1.
If a variable $x$ has any Normal distribution $N(\mu, \sigma)$ with mean $\mu$ and standard deviation $\sigma$, then the standardized variable

$$
z=\frac{x-\mu}{\sigma}
$$

has the standard Normal distribution, $N(0,1)$.


## The Standard Normal Table

Because all Normal distributions are the same when we standardize, we can find areas under any Normal curve from a single table.

## Definition: The Standard Normal Table (Appendix A - p. 762)

Table 2 is a table of areas under the standard Normal curve. The table entry for each value $z$ is the area under the curve to the left of $z$.

Suppose we want to find the proportion of observations from the standard Normal distribution that are less than 0.81 .
We can use $Z$ table ( $z^{*}$ ):

| $\boldsymbol{Z}$ | .00 | . $\mathbf{0}^{+}$ | .02 |
| :---: | :---: | :---: | :---: |
| $\mathbf{0 . 7}$ | .7580 | .7 | .7642 |
| $\mathbf{0 . 8}$ | .7881 | .7910 | .7939 |
| $\mathbf{0 . 9}$ | .8159 | .8186 | .8212 |

## $\mathrm{P}(\mathrm{z}<0.81)=.7910$



## The Standard Normal Table

Because all Normal distributions are the same when we standardize, we can find areas under any Normal curve from a single table.

## Definition: The Standard Normal Table (Appendix A - p. 762)

Table 2 is a table of areas under the standard Normal curve. The table entry for each value $z$ is the area under the curve to the left of $z$.

Suppose we want to find the proportion of observations from the standard Normal distribution that are less than -1.62.
We can use $Z$ table $\left(z^{*}\right)$ :

| $\boldsymbol{Z}$ | .00 | .01 | .02 |
| :---: | :---: | :---: | :---: |
| -1.7 | .0446 | .0436 | .0427 |
| -1.6 | .0548 | .0537 | .0526 |
| -1.5 | .0668 | .0655 | .0643 |

$$
P(z<-1.62)=.0526
$$



Finding Areas Under the Standard Normal Curve
Find the proportion of observations from the standard Normal distribution that are between -1.25 and 0.81 .

## Example

## Finding Areas Under the Standard Normal Curve

Find the proportion of observations from the standard Normal distribution that are between -1.25 and 0.81 .


Area to left of
$z=-1.25$ is 0.1056 .


Area between $z=-1.25$ and $z=0.81$ is $0.7910-0.1056=0.6854$.


Can you find the same proportion using a different approach?


$$
\begin{aligned}
& 1-(0.1056+0.2090)= \\
& 1-0.3146=0.6854
\end{aligned}
$$

## Normal Distribution Calculations

How to Solve Problems Involving Normal Distributions

State: Express the problem in terms of the observed variable $x$.
Plan: Draw a picture of the distribution and shade the area of interest under the curve.

Do: Perform calculations.
-Standardize $x$ to restate the problem in terms of a standard Normal variable $z$.

- Use Standard Table and the fact that the total area under the curve is 1 to find the required area under the standard Normal curve.

Conclude: Write your conclusion in the context of the problem.

## Normal Distribution Calculations

When Tiger Woods hits his driver, the distance the ball travels can be described by $N(304,8)$. What percent of Tiger's drives travel between 305 and 325 yards?


Using Table A, we can find the area to the left of $z=2.63$ and the area to the left of $z=0.13$. $0.9957-0.5517=0.4440$. About 44\% of Tiger's drives travel between 305 and 325 yards.

1) Given a fairly normal distribution that has a mean of 100 and a standard deviation of 15 , what are the following $z$-scores:
A) for an observation that is 110
B) for an observation that is 85
C) for an observation that is 142
2) Using the information from the problem above, what is the value of the observation that has a z-score:
a) $z=-2$; b) $z=1.58$
3) Given a fairly symmetric distribution that has a mean of 100 and a standard deviation of 15, what are the following z-scores:

$$
\text { z score }=\frac{x-\text { mean }}{\text { standard deviation }}
$$

A) for an observation that is $110110-100$

$$
z \text { score }=\frac{110-100}{15}=0.667
$$

B) for an observation that is 85
C) for an observation that is 142

$$
z \text { score }=\frac{85-100}{15}=-1.0
$$

$$
z \text { score }=\frac{142-100}{15}=2.8
$$

## Friday, Sept: Warm-UP

2) Using the information from the problem above, what is the value of the observation that has a z-score: a) $z$ score $=-2$

$$
-2=\frac{x-100}{15} \quad-30=x-100
$$

b) z score $=1.58$

$$
1.58=\frac{x-100}{15} \quad \begin{gathered}
23.7=x-100 \\
\therefore x=123.7
\end{gathered}
$$

Standard Deviation Activity (class activity)

- Pick up the worksheet and complete all the problems \#1 to 3 without using a calculator. Use your mental math and estimation skills.
- Be prepared to share your results.
- Wait for instructions...


## Chebyshev's Rule

- Given any distribution, you can estimate a minimum percentage of observations that will fall within some number ( $k$ ) of standard deviations from the mean, using the following formula:
- $k=$ some number of standard deviations
$\square$ Percentage of observations: $1-\frac{1}{k^{2}}$


## HW Ques: Section 4.2 (p. 171)

\#4.31 a) Calculate the $\bar{x}$ and $s_{x}$ for both samples: Sample $1 \bar{x}=7.81 s_{x}=0.3985$

$$
\text { Sample } 2 \bar{x}=49.68 \quad s_{x}=1.7390
$$



\#4.31 b) 5.1 and 3.5
Huh? 5.1 and 3.5 what?? NO context! (-1 pt)

## HW Ques: Section 4.2 (p. 171)

\#4.31 b) Calculate the coefficient of variation coefficient of variation is: $C V=100 \frac{s}{\bar{x}}$

$$
\text { Sample } 1 \mathrm{CV}=100 \frac{0.3985}{7.81}=5.102
$$

$$
\text { Sample } 2 \mathrm{CV}=100 \frac{1.739}{49.68}=3.50
$$

## Warm-Up <br> Assessing Normality

- Example 1: Given that the average heights of NBA players is $6^{\prime} 6^{\prime \prime}$ with a standard deviation of 3 inches, determine if the distribution is normal based upon a group with the tallest player being $7^{\prime} 1^{\prime \prime}$, and the shortest player being 6'4" (...still today?)
- Example 2: (p. 184 Problem \#4.38). Average playing time of compact discs. Answer parts $a, b$, and $c$


## Assessing Normality

The Normal distributions provide good models for some distributions of real data. Many statistical inference procedures are based on the assumption that the population is approximately Normally distributed. Consequently, we need a strategy for assessing Normality.
$\checkmark$ Plot the data.
-Make a dotplot, stemplot, or histogram and see if the graph is approximately symmetric and bell-shaped.
$\checkmark$ Check whether the data follow the 68-95-99.7 rule.
-Count how many observations fall within one, two, and three standard deviations of the mean and check to see if these percents are close to the $68 \%, 95 \%$, and $99.7 \%$ targets for a Normal distribution.

## Assessing Normality

- Example 1: Given that the average heights of NBA players is $6^{\prime} 6^{\prime \prime}$ with a standard deviation of 3 inches, determine if the distribution is normal based upon a group with the tallest player being $7^{\prime} 1^{\prime \prime}$, and the shortest player being 6 ' 4 ".
- Use the Empirical Rule: 68-95-99.7
- Draw a normal density curve and determine the values for heights that are $\pm 1 \sigma$, $\pm 2 \sigma$, and $\pm 3 \sigma$ away from the mean


## Assessing Normality

Draw a normal density curve and determine the values for heights that are $\pm 1 \sigma, \pm 2 \sigma$, and $\pm 3 \sigma$ away from the mean


## The Empirical Rule vs. Chebyshev's Rule

- The Empirical Rule can only be used for distributions that are Normal, or approx. Normal
- Given any distribution, you can use Chebyshev’s Rule to estimate a minimum percentage of observations that will fall within some number ( $k$ ) of standard deviations from the mean, using the following formula:
- $k=$ some number of standard deviations
- Percentage of observations: $1-\frac{1}{k^{2}}$

Assessing Normality Example 2: (p.184) \#4.38: Average playing time of compact discs
A) $\bar{x}=35 \mathrm{~min} . s_{x}=5$ What is $\pm \sigma$ ?

$$
\begin{aligned}
& -1 \sigma=30 \mathrm{~min} ;+1 \sigma=40 \mathrm{~min} \\
& -2 \sigma=25 \mathrm{~min} ;+2 \sigma=45 \mathrm{~min}
\end{aligned}
$$

- B) At least what \% of times are between 25 and 45 min.? (use Chebyshev's Rule)

$$
\begin{aligned}
& 1-\frac{1}{k^{2}} \rightarrow u \operatorname{sing} a \sigma=2 \\
& 1-\frac{1}{(2)^{2}}=1-\frac{1}{4}=0.75
\end{aligned}
$$

$\therefore$ we can estimate at least $75 \%$ of the CD's will fall b/t 25 and 45 min.

## Section 4.5 Normal Distributions

## Summary

In this section, we learned that...
$\checkmark$ The Normal Distributions are described by a special family of bellshaped, symmetric density curves called Normal curves. The mean $\mu$ and standard deviation $\sigma$ completely specify a Normal distribution $N(\mu, \sigma)$. The mean is the center of the curve, and $\sigma$ is the distance from $\mu$ to the change-of-curvature points on either side.
$\checkmark$ All Normal distributions obey the 68-95-99.7 Rule, which describes what percent of observations lie within one, two, and three standard deviations of the mean.

## Summary of Normal Distributions

## Summary

In this section, we learned that...
$\checkmark$ All Normal distributions are the same when measurements are standardized. The standard Normal distribution has mean $\mu=0$ and standard deviation $\sigma=1$.
$\checkmark$ Standard Normal Table gives percentiles for the standard Normal curve. By standardizing, we can use Table A to determine the percentile for a given $z$-score or the $z$-score corresponding to a given percentile in any Normal distribution.
$\checkmark$ To assess Normality for a given set of data, we first observe its shape. We then check how well the data fits the 68-95-99.7 rule. We can also construct and interpret a Normal probability plot.

## Looking Ahead...

## In Chapter 5...

We'll learn how to describe relationships between two quantitative variables

We'll study
$\checkmark$ Bivariate relationships
$\checkmark$ Scatterplots and correlation
$\checkmark$ Least-squares regression lines (LSRL)
$\checkmark$ Residuals \& residual plots

## Sept 26, 2017

Warm Up: Self Reflection
Complete the 1 st 6 -Weeks Reflection Paper (both sides)

