Stats & Probability
Chapter 6

Probability

6.1: Chance Experiment & Sample Space

A **chance experiment** is any activity or situation in which there is uncertainty about which of two or more possible outcomes will result (Not really a scientific research experiment, but an experiment non the less…).

The collection of all possible outcomes of a chance experiment is the **sample space** for the experiment.

Example

An experiment is to be performed to study student preferences in the food line in the cafeteria. Specifically, the staff wants to analyze the effect of the student's gender on the preferred food line (burger, salad or main entrée).

Example - continued

The sample space consists of the following six possible outcomes.

1. A male choosing the burger line.
2. A female choosing the burger line.
3. A male choosing the salad line.
4. A female choosing the salad line.
5. A male choosing the main entrée line.
6. A female choosing the main entrée line.

(If order is not important to the situation, could be said burger line chosen by a male, etc…)

Example - continued

The sample space could be represented by using set notation and ordered pairs.

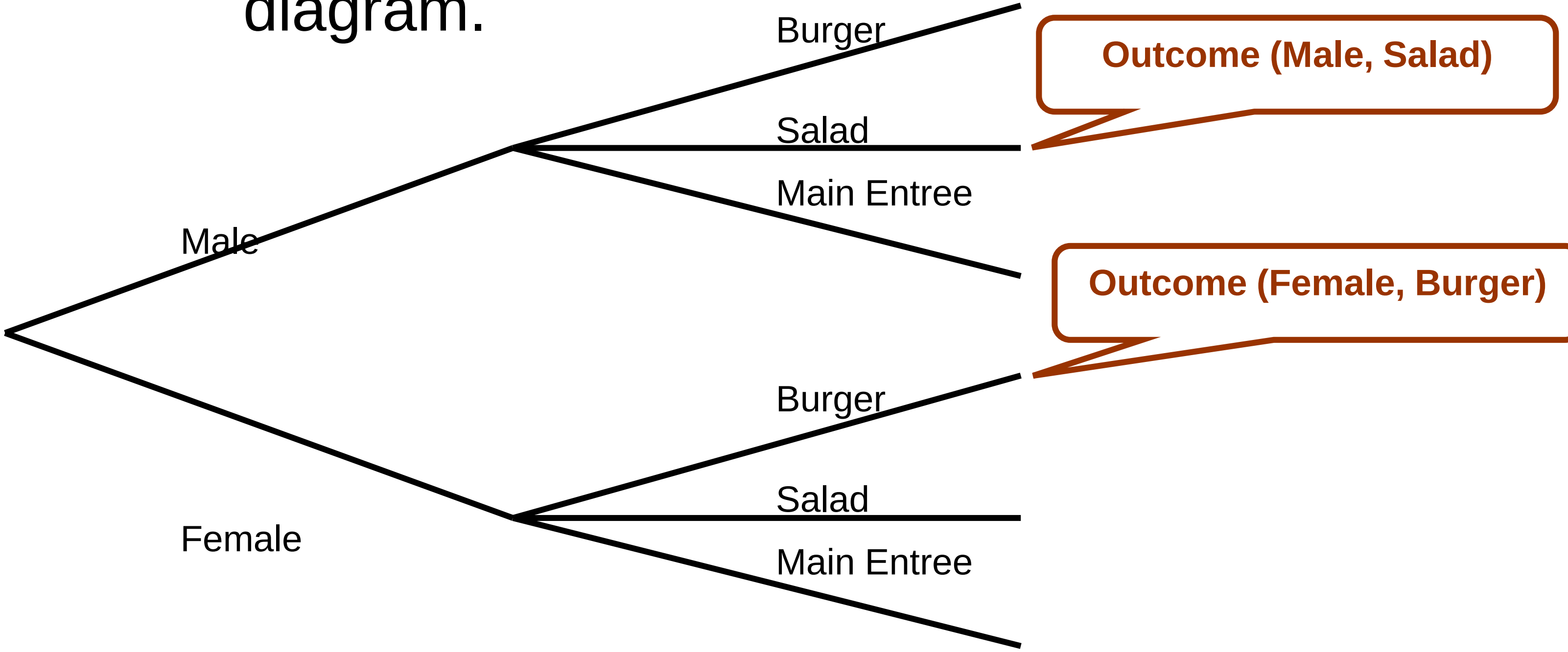
sample space = {(male, burger), (female, burger), (male, salad), (female, salad), (male, main entree), (female, main entree)}

If we use M to stand for male, F for female, B for burger, S for salad and E for main entrée the notation could be simplified to

sample space = {MB, FB, MS, FS, ME, FE}

Example - continued

Yet another way of illustrating the sample space would be using a picture called a "tree" diagram.



This "tree" has two sets of "branches" corresponding to the two bits of information gathered. To identify any particular outcome of the sample space, you traverse the tree by first selecting a branch corresponding to gender and then a branch corresponding to the choice of food line. (If order is not important to the situation, burger, salad & entree could be 1st branch)

Events

An **event** is any collection of outcomes from the sample space of a chance experiment.

A **simple event** is an event consisting of exactly one outcome.

If we look at the lunch line example and use the following sample space description {MB, FB, MS, FS, ME, FE}

The event that the student selected is male is given by

male = {MB, MS, ME}

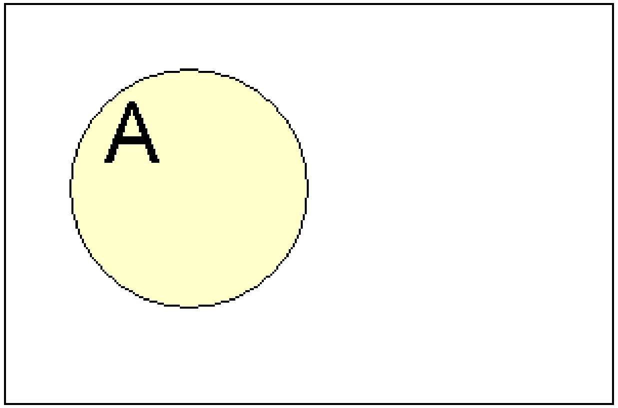
The event that the preferred food line is the burger line is given by burger = {MB, FB}

The event that the person selected is a female that prefers the salad line is {FS}. This is an example of a simple event & there are 6 possible simple events that could occur).

Venn Diagrams

A **Venn Diagram** is an informal picture that is used to identify relationships.

The collection of all possible outcomes of a chance experiment are represented as the interior of a rectangle.

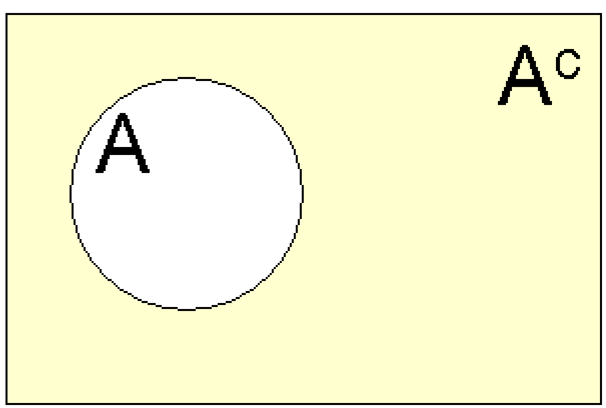


The rectangle represents the sample space and shaded area represents the event A.

Forming New Events

Let A and B denote two events.

The event **not A** consists of all experimental outcomes that are not in event A. Not A is sometimes called the **complement** of A and is usually denoted by A^c, A', C(A), S - A , not A, -A or possibly $\overline{A}$.



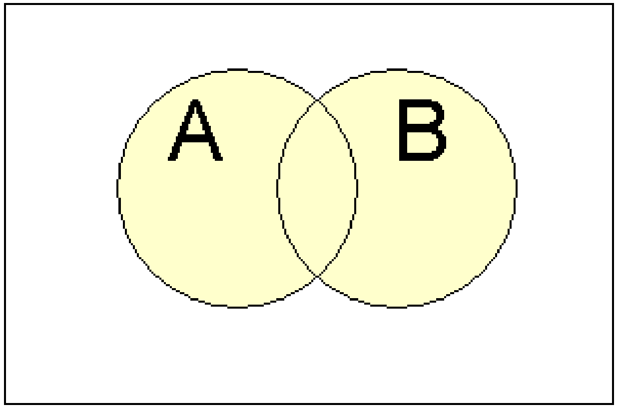
The shaded area represents the event not A.

Forming New Events

Let A and B denote two events.

The event **A or B** consists of all experimental outcomes that are in at least one of the two events, that is, in A or in B or in both of these. **A or B** is called the **union** of the two events and is denoted by

$\mathbf{A \cup B}$

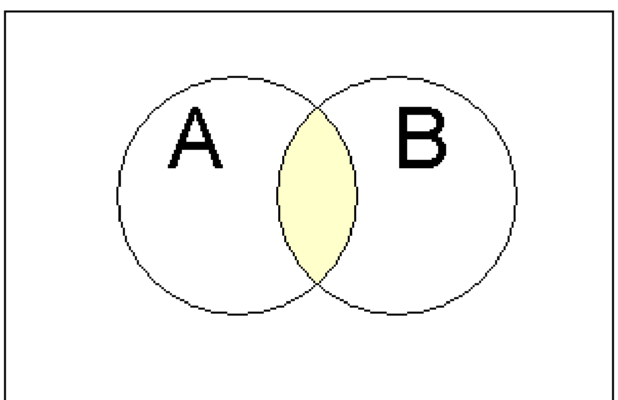


The shaded area represents the event $A \cup B$.

Forming New Events

Let A and B denote two events.

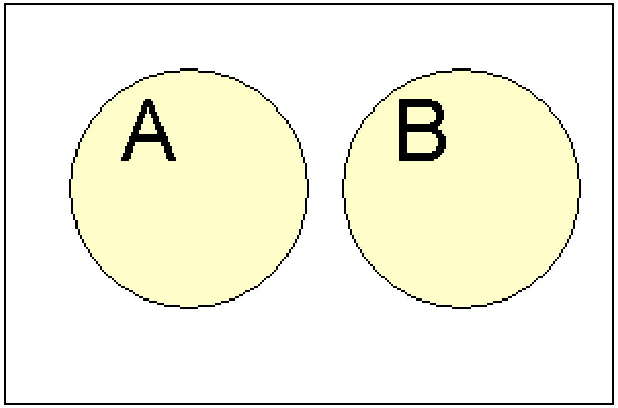
The event **A and B** consists of all experimental outcomes that are in both of the events A and B. **A and B** is called the **intersection** of the two events and is denoted by $\mathbf{A \cap B}$



The shaded area represents the event $A \cap B$.

More on intersections

Two events that have no common outcomes are said to be **disjoint** or **mutually exclusive**.



A and B are disjoint events

More than 2 events

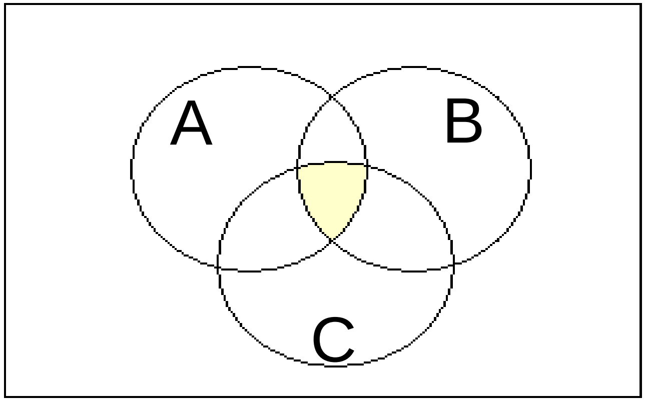
Let A_1, A_2, …, A_k denote k events

The events A_1 or A_2 or … or A_k consist of all outcomes in at least one of the individual events. [i.e. A_1, A_2, …, A_k]

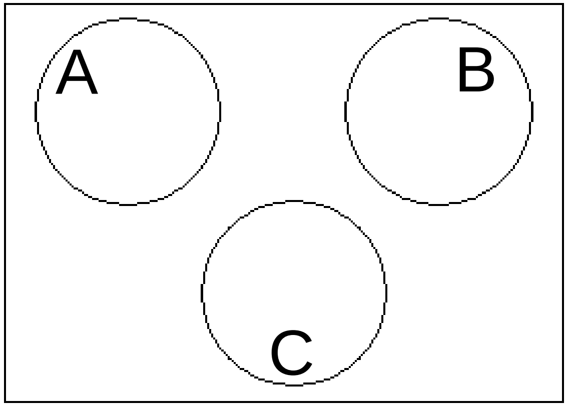
The events A_1 and A_2 and … and A_k consist of all outcomes that are simultaneously in every one of the individual events.
[i.e. A_1, A_2, …, A_k]

These k events are disjoint if no two of them have any common outcomes.

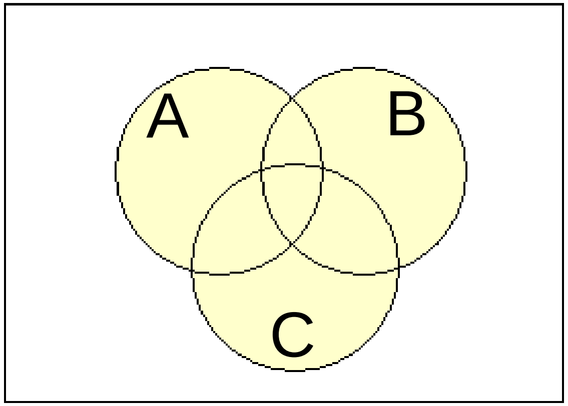
Some illustrations



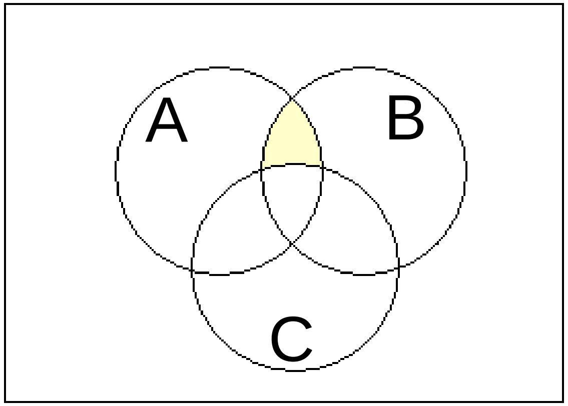
$A \cap B \cap C$



A, B & C are Disjoint



$A \cup B \cup C$



$A \cap B$ NOT C

Mutually Exclusive vs. Independent

It’s common to confuse the concepts of ME and Indep.

If A happens, then event B cannot, or vice-versa. The two events "it rained on Tuesday" and "it did not rain on Tuesday" are mutually exclusive events. When calculating the probabilities for ME events you add the probabilities.   With respect to independence, the outcome of event A, has no effect on the outcome of event B. Such as "It rained on Tuesday" and "My chair broke at work". When calculating the probabilities for independent events you multiply the probabilities. You are effectively saying what is the chance of both events happening bearing in mind that the two were unrelated.

Mutually Exclusive vs. Independent cont…

So, if A and B are mutually exclusive, they cannot be independent. If A and B are independent, they cannot be mutually exclusive. However, If the events were it rained today" and "I left my umbrella at home" they are not mutually exclusive, but they are probably not independent either, because one would think that you'd be less likely to leave your umbrella at home on days when it rains.

Mutually Exclusive example

What happens if I have 1 die & want to throw 1 and 6 in any order? This now means that we do not mind if the first die is either 1 or 6, as we are still in with a chance. But with the first die, if 1 falls uppermost, clearly it rules out the possibility of 6 being uppermost, so the two Outcomes, 1 and 6, are exclusive. One result directly affects the other. In this case, the probability of throwing 1 or 6 with the <u>first die</u> is the sum of the two probabilities, 1/6 + 1/6 = 1/3.

The probability of the second die being favorable is still 1/6 as the second die can only be one specific number, a 6 if the first die is 1, and vice versa.

Therefore the probability of throwing 1 and 6 in any order with one die thrown twice is 1/3 x 1/6 = 1/18. Note that we multiplied the last two probabilities as they were independent of each other!!!

Independent example

Now with 2 dice, what is the probability of throwing a one & a six is the result of throwing one with the first die and six with the second die (or visa versa). The total possibilities are, one from six outcomes for the first event and one from six outcomes for the second, Therefore (1/6) * (1/6) = 1/36th or 2.77%. Since order didn't matter (1,6 or 6,1) it's $2/36^{th}$ as there 2 ways to get it.

The two events are independent, since whatever happens to the first die cannot affect the throw of the second, the probabilities are therefore multiplied, and remain 1/18th. Same P, but different way to calculate it.

Actually, this is the P(any pair) with 2 die.

6.2: Probability – Classical Approach

If a chance experiment has k outcomes, all equally likely, then each individual outcome has the probability 1/k and the probability of an event E is

$$P(E) = \frac{\text{number of outcomes favorable to E}}{\text{number of outcomes in the sample space}}$$

Probability - Example

Consider the experiment consisting of rolling two fair dice and observing the sum of the up faces. A sample space description is given by

$$\{(1, 1), (1, 2), \ldots, (6, 6)\}$$

where the pair (1, 2) means 1 is the up face of the 1st die and 2 is the up face of the 2nd die. This sample space consists of 36 equally likely outcomes.

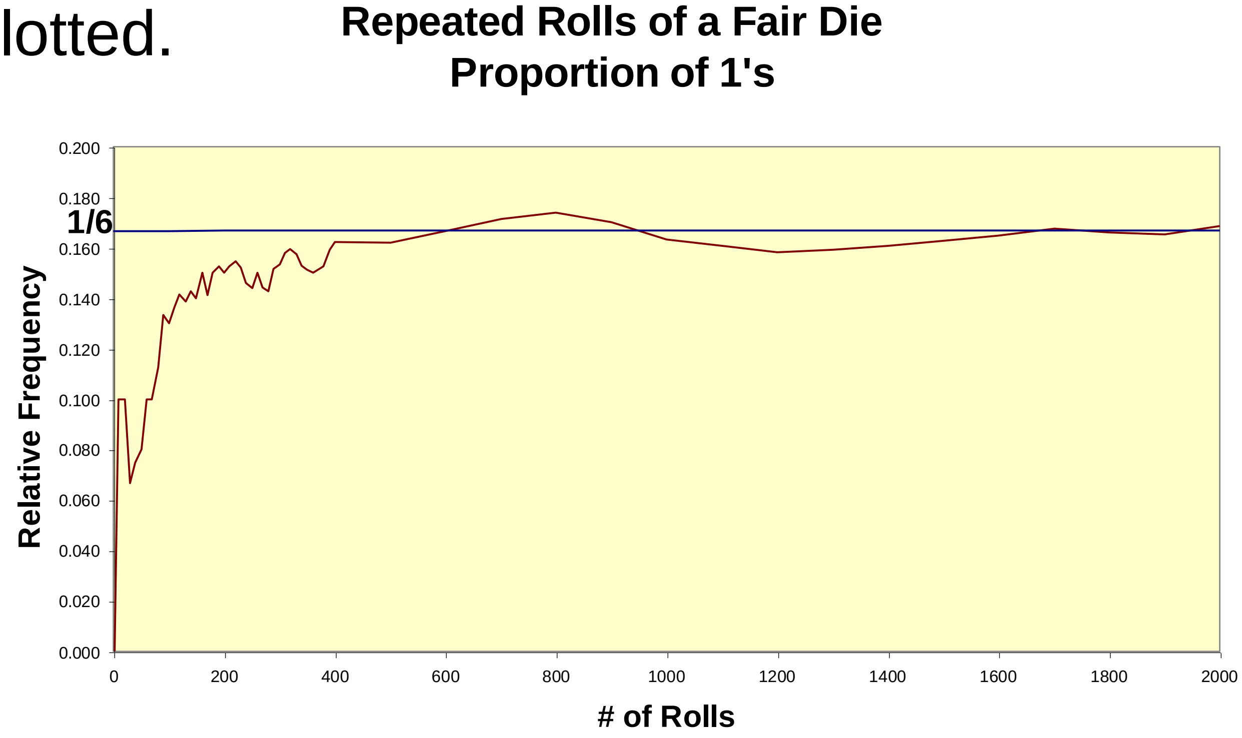
Let E stand for the event that the sum is 6.

Event E is given by E={(1, 5), (2, 4), (3, 3), (4, 2), (5, 1)}. The event consists of 5 outcomes, so

$$P(E) = \frac{5}{36} = 0.1389$$

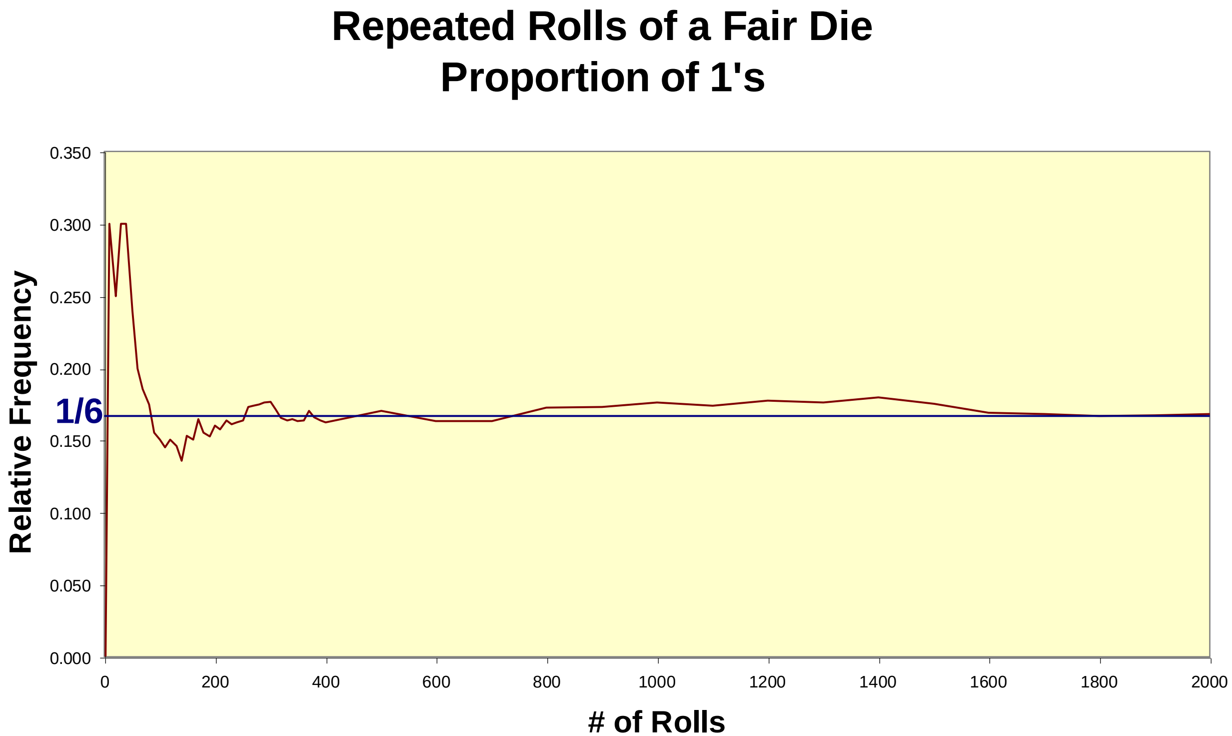
Probability - Empirical Approach

Consider the chance experiment of rolling a "fair" die. We would like to investigate the probability of getting a "1" for the up face of the die. The die was rolled and after each roll the up face was recorded and then the proportion of times that a 1 turned up was calculated and plotted.



Probability - Empirical Approach

The process was simulated again and this time the result were similar. Notice that the proportion of 1’s seems to stabilize and in the long run gets closer to the “theoretical” value of 1/6.



Probability - Empirical Approach

In many "real-life" processes and chance experiments, the probability of a certain outcome or event is unknown, but never the less this probability can be estimated reasonably well from observation. The justification if the Law of Large Numbers.

Law of Large Numbers: As the number of repetitions of a chance experiment increases, the chance that the relative frequency of occurrence for an event will differ from the true probability of the event by more than any very small number approaches zero.

Relative Frequency Approach

The **probability of an event E**, denoted by **P(E)**, is defined to be the value approached by the relative frequency of occurrence of E in a very long series of trials of a chance experiment. Thus, if the number of trials is quite large,

$$P(E) \approx \frac{\text{number of times E occurs}}{\text{number of trials}}$$

Methods for Determining Probability

1. **The classical approach:** Appropriate for experiments that can be described with equally likely outcomes.
2. **The subjective approach:** Probabilities represent an individual's judgment based on facts combined with personal evaluation of other information.
3. **The relative frequency approach:** An estimate is based on an accumulation of experimental results. This estimate, usually derived empirically, presumes a replicable chance experiment.

6.3: Basic Properties of Probability

1. For any event E, $0 \le P(E) \le 1$.

2. If S is the sample space for an experiment, $P(S)=1$.

3. If two events E and F are disjoint, then $P(E \text{ or } F) = P(E) + P(F)$.

4. For any event E,
$$P(E) + P(\text{not } E) = 1 \text{ so,}$$
$P(\text{not } E) = 1 – P(E)$ and $P(E) = 1 – P(\text{not } E)$.

Equally Likely Outcomes

Consider an experiment that can result in any one of N possible outcomes. Denote the corresponding simple events by O_1, O_2,… O_n. If these simple events are equally likely to occur, then

$$1.\ P(O_1) = \frac{1}{N}, P(O_2) = \frac{1}{N}, \cdots, P(O_N) = \frac{1}{N}$$

2. For any event E,

$$P(E) = \frac{\text{number of outcomes in E}}{N}$$

Example

Consider the experiment consisting of randomly picking a card from an ordinary deck of playing cards (52 card deck).

Let A stand for the event that the card chosen is a King.

The sample space is given by S = {A♠, K♠,…,2♠, A♥ , K ♥ , …, 2♥, A♦,…, 2♦, A♣,…, 2♣} and consists of 52 equally likely outcomes.

The event is given by

A={K♣, K♦, K♥, K♠}

and consists of 4 outcomes, so

$$P(A) = \frac{4}{52} = \frac{1}{13} = 0.0769$$

Example

Consider the experiment consisting of rolling two fair dice and observing the sum of the up faces.
Let E stand for the event that the sum is 7.

The sample space is given by

$$\mathbf{S}=\{(1\ ,1),\ (1,\ 2),\ \ldots\ ,\ (6,\ 6)\}$$

and consists of 36 equally likely outcomes.

The event E is given by

$$\mathbf{E}=\{(1,\ 6),\ (2,\ 5),\ (3,\ 4),\ (4,\ 3),\ (5,\ 2),\ (6,\ 1)\}$$

and consists of 6 outcomes, so

$$P(E) = \frac{6}{36}$$

Example

Consider the experiment consisting of rolling two fair dice and observing the sum of the up faces.
Let F stand for the event that the sum is 11.

The sample space is given by

$$\mathbf{S}=\{(1\ ,1), (1, 2), \ldots , (6, 6)\}$$

and consists of 36 equally likely outcomes.

The event F is given by

$$\mathbf{F}=\{(5, 6), (6, 5)\}$$

and consists of 2 outcomes, so

$$P(F) = \frac{2}{36}$$

Warm-UP

1) What does it mean for two events to be mutually exclusive?
2) What does it mean for two events to be independent?
3) Can events be both mutually exclusive and independent?
4) A and B are independent events, and P(A) = 0.7, while the P(B) = 0.3. What is:
    a) P(B | A)
    b) $P(B \cap A)$

Warm-UP

1) What does it mean for two events to be mutually exclusive?

Two events are ***mutually exclusive*** or ***disjoint*** when they share no common outcomes. The complement of an event is always mutually exclusive from the event.

Ex: If I roll a die, the events A: rolling an odd number ; and B: rolling an even number are mutually exclusive.

Ex: If I roll a die, the events A: rolling an odd number ; and B: rolling a prime number

2) What does it mean for two events to be independent?

6.5: Independence

Two events E and F are said to be **independent** if the occurrence of one event does not effect the occurrence of the other event, and vice versa. When this is true, then $P(E \mid F) = P(E)$ and $P(F \mid E) = P(F)$.

Conversely, if events E and F are subsets of the same sample space, and they are not independent, they are said to be **dependent** events.

Oct 28/29 Warm-UP

1) Can events be both mutually exclusive and independent?
2) A and B are independent events, and P(A) = 0.6, while the P(B) = 0.4. What is:
   a) P(B | A) b) P $(B \cap A)$ c) P $(B \cup A)$
3) An event and its complement are ___________ mutually exclusive. Always Sometimes Never
4) Mutually exclusive events are ______________ complements of each other.

   Always Sometimes Never

Warm-UP

1) Can events be both mutually exclusive and independent? **No** (essentially no), because we typically refer to mutually exclusive events as subsets from the same sample space, whereas independent events are events from separate sample spaces

2) A and B are independent events, and P(A) = 0.6, while the P(B) = 0.4. What is:

   a) P(B | A) = **0.4**

   b) $P(B \cap A) = 0.6 \cdot 0.4$
   $= 0.24$

Warm-UP

3) An event and its complement are ___________ mutually exclusive.

Always Sometimes Never

3) Mutually exclusive events are _____________ complements of each other.

Always Sometimes Never

Addition Rule for Disjoint Events

Let E and F be two disjoint events.

One of the basic properties of probability is,
$P(E \text{ or } F) = P(E \cup F) = P(E) + P(F)$

More Generally, if $E_1, E_2, \ldots, E_k$ are disjoint, then
$P(E_1 \text{ or } E_2 \text{ or } \ldots \text{ or } E_k) = P(E_1 \cup E_2 \cup \ldots \cup E_k)$
$= P(E_1) + P(E_2) \ldots + P(E_k)$

$P(E_1 \cup E_2 \cup \ldots \cup E_k)$

Example

Consider the experiment consisting of rolling two fair dice and observing the sum of the up faces. Let E stand for the event that the sum is 7 and F stand for the event that the sum is 11.

$$P(E) = \frac{6}{36} \text{ \& } P(F) = \frac{2}{36}$$

Since E and F are disjoint events

$$P(E \cup F) = P(E) + P(F) = \frac{6}{36} + \frac{2}{36} = \frac{8}{36}$$

A Leading Example

A study[1] was performed to look at the relationship between motion sickness and seat position in a bus. The following table summarizes the data.

	Seat Position in Bus		
	Front	Middle	Back
Nausea	58	166	193
No Nausea	870	1163	806

[1] "Motion Sickness in Public Road Transport: The Effect of Driver, Route and Vehicle" (*Ergonomics* (1999): 1646 – 1664).

A Leading Example

Let's use the symbols N, N^C, F, M, B to stand for the events Nausea, No Nausea, Front, Middle and Back respectively.

	Seat Position in Bus			
	Front	Middle	Back	Total
Nausea	58	166	193	417
No Nausea	870	1163	806	2839
Total	928	1329	999	3256

A Leading Example

Computing the probability that an individual in the study gets nausea we have

$$P(N) = \frac{417}{3256} = 0.128$$

	Seat Position in Bus			
	Front	Middle	Back	Total
Nausea	58	166	193	417
No Nausea	870	1163	806	2839
Total	928	1329	999	3256

A Leading Example

Other probabilities are easily calculated by dividing the numbers in the cells by 3256 to get

	Seat Position in Bus			
	Front	Middle	Back	Total
Nausea	0.018	0.051	0.059	0.128
No Nausea	0.267	0.357	0.248	0.872
Total	0.285	0.408	0.307	1.000

P(N and F)

P(N)

P(F)

P(M and N^C)

A Leading Example

The event “a person got nausea given he/she sat in the front seat” is an example of what is called a conditional probability.

Of the 928 people who sat in the front, 58 got nausea so the probability that “a person got nausea given he/she sat in the front seat is

$$\frac{58}{928} = 0.0625$$

6.4: Conditional Probability

If we want to see if nausea is related to seat position we might want to calculate the probability that “a person got nausea given he/she sat in the front seat.”

We usually indicate such a conditional probability with the notation $P(N \mid F)$.

$P(N \mid F)$ stands for the “probability of N given F.

Conditional Probability

Let E and F be two events with P(F) > 0. The **conditional probability of the event E given that the event F has occurred**, denoted by **P(E|F)**, is

$$P(E\,|\,F) = \frac{P(E \cap F)}{P(F)}$$

6.5: Independence

Two events E and F are said to be **independent** if the occurrence of one event does not effect the occurrence of the other event, and vice versa. When this is true, then $P(E \mid F) = P(E)$ and $P(F \mid E) = P(F)$.

If E and F are not independent, they are said to be **dependent** events.

If $P(E|F) = P(E)$, it is also true that $P(F|E) = P(F)$ and vice versa. So if E is independent of F, F is independent of E.

Example

A survey of job satisfaction[2] of teachers was taken, giving the following results

		Job Satisfaction		
		Satisfied	Unsatisfied	Total
LEVEL	College	74	43	117
	High School	224	171	395
	Elementary	126	140	266
	Total	424	354	778

[2] "Psychology of the Scientist: Work Related Attitudes of U.S. Scientists" (*Psychological Reports* (1991): 443 – 450).

Example

If all the cells are divided by the total number surveyed, 778, the resulting table is a table of empirically derived probabilities.

		Job Satisfaction		
		Satisfied	Unsatisfied	Total
LEVEL	College	0.095	0.055	0.150
	High School	0.288	0.220	0.508
	Elementary	0.162	0.180	0.342
	Total	0.545	0.455	1.000

Example

		Job Satisfaction		
		Satisfied	Unsatisfied	Total
LEVEL	College	0.095	0.055	0.150
	High School	0.288	0.220	0.508
	Elementary	0.162	0.180	0.342
	Total	0.545	0.455	1.000

For convenience, let C stand for the event that the teacher teaches college, S stand for the teacher being satisfied and so on. Let's look at some probabilities and what they mean.

$P(C) = 0.150$ is the proportion of teachers who are college teachers.

$P(S) = 0.545$ is the proportion of teachers who are satisfied with their job.

$P(C \cap S) = 0.095$ is the proportion of teachers who are college teachers and who are satisfied with their job.

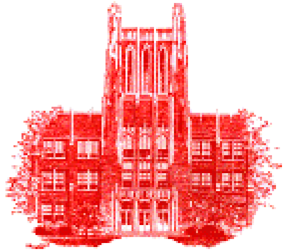
Example

		Job Satisfaction		
		Satisfied	Unsatisfied	Total
LEVEL	College	0.095	0.055	0.150
	High School	0.288	0.220	0.508
	Elementary	0.162	0.180	0.342
	Total	0.545	0.455	1.000

The proportion of teachers who are college teachers given they are satisfied is

$$P(C \mid S) = \frac{P(C \cap S)}{P(S)} = \frac{0.095}{0.545} = 0.175$$

Restated: This is the proportion of satisfied that are college teachers.



Example

		Job Satisfaction		
		Satisfied	Unsatisfied	Total
LEVEL	College	0.095	0.055	0.150
	High School	0.288	0.220	0.508
	Elementary	0.162	0.180	0.342
	Total	0.545	0.455	1.000

What is the proportion of teachers who are satisfied given they are college teachers?

$$P(S \mid C) = \frac{P(S \cap C)}{P(C)} = \frac{P(C \cap S)}{P(C)}$$

$$= \frac{0.095}{0.150} = 0.632$$

Restated: This is the proportion of college teachers that are satisfied.

Example

		Job Satisfaction		
		Satisfied	Unsatisfied	Total
LEVEL	College	0.095	0.055	0.150
	High School	0.288	0.220	0.508
	Elementary	0.162	0.180	0.342
	Total	0.545	0.455	1.000

What is P (C) = ?   What is P (C | S) = ?

$$P(C) = 0.150 \text{ and } P(C \mid S) = \frac{P(C \cap S)}{P(S)} = \frac{0.095}{0.545} = 0.175$$

$\boxed{P(C|S) \neq P(C) \text{ so C and S are dependent events.}}$

Multiplication Rule for Independent Events

The events E and F are independent if and only if $P(E \cap F) = P(E) \cdot P(F)$

That is, independence implies the relation $P(E \cap F) = P(E) \cdot P(F)$, and this relation implies independence.

Example

Consider the person who purchases from two different manufacturers a TV and a DVD. Suppose we define the events A and B by

A = event the TV doesn't work properly

B = event the DVD doesn't work properly

Suppose P(A) = 0.01 and P(B) = 0.02.

If we assume that the events A and B are independent[3], then

$$P(A \text{ and } B) = P(A \cap B) = (0.01)(0.02) = 0.0002$$

[3] This assumption seems to be a reasonable assumption since the manufacturers are different.

Example

Consider the teacher satisfaction survey

		Job Satisfaction		
		Satisfied	Unsatisfied	Total
LEVEL	College	0.095	0.055	0.150
	High School	0.288	0.220	0.508
	Elementary	0.162	0.180	0.340
	Total	0.545	0.455	1.000

$P(C) = 0.150$, $P(S) = 0.545$ and $P(C \cap S) = 0.095$

Since $P(C)P(S) = (0.150)(0.545) = 0.08175$ and $P(C \cap S) = 0.095$, $P(C \cap S) \neq P(C)P(S)$. This shows that C & S are dependent events.

Sampling Schemes

Sampling is **with replacement** if, once selected, an individual or object is put back into the population before the next selection. These would represent **independent** **events.**

Sampling is **without replacement** if, once selected, an individual or object is not returned to the population prior to subsequent selections. These would represent **dependent** **events.**

Example

Suppose we are going to select three cards from an ordinary deck of cards. Consider the events:

E_1 = event that the first card is a king

E_2 = event that the second card is a king

E_3 = event that the third card is a king.

Example – With Replacement

If we select the first card and then place it back in the deck before we select the second, and so on, the sampling will be with replacement.

$$P(E_1) = P(E_2) = P(E_3) = \frac{4}{52}$$

$$P(E_1 \cap E_2 \cap E_3) = P(E_1)P(E_2)P(E_3)$$

$$= \frac{4}{52}\,\frac{4}{52}\,\frac{4}{52} = 0.000455$$

Example – Without Replacement

If we select the cards in the usual manner without replacing them in the deck, the sampling will be without replacement.

$$P(E_1) = \frac{4}{52}, \; P(E_2) = \frac{3}{51}, \; P(E_3) = \frac{2}{50}$$

$$P(E_1 \cap E_2 \cap E_3) = P(E_1)P(E_2)P(E_3)$$
$$= \frac{4}{52}\,\frac{3}{51}\,\frac{2}{50} = 0.000181$$

Example: Jury Selection

Suppose the attorneys hope for a representative jury. If a jury pool in a city contains 12,000 potential jurors and 3000 of them are Hispanic.

Consider the defined events

E_1 = event that the first juror selected is Hispanic

E_2 = event that the second juror selected is Hispanic

E_3 = event that the third juror selected is Hispanic

E_4 = event that the forth juror selected is Hispanic

Example: Jury Selection

Clearly the sampling will be without replacement so

$$P(E_1) = \frac{3000}{12000}, \; P(E_2) = \frac{2999}{11999},$$

$$P(E_3) = \frac{2998}{11998}, P(E_4) = \frac{2997}{11997}$$

So $P(E_1 \cap E_2 \cap E_3 \cap E_4)$

$$= \frac{3000}{12000}\,\frac{2999}{11999}\,\frac{2998}{11998}\,\frac{2997}{11997} = 0.003900$$

A Practical Example - continued

If we "treat" the Events E_1, E_2, E_3 and E_4 as being with replacement (independent) we would get

$$P(E_1) = P(E_2) = P(E_3) = \frac{3000}{12000} = 0.25$$

$$\text{So } P(E_1 \cap E_2 \cap E_3 \cap E_4) = (0.25)(0.25)(0.25)(0.25)$$

$$= 0.003906$$

A Practical Example - continued

Notice the result calculate by sampling **without replacement** is **0.003900** and the result calculated by sampling **with replacement** is **0.003906**. These results are essentially the same.

So, when the number of items in the population is large and the number in the sample is small, the two methods give essentially the same result.

An Observation

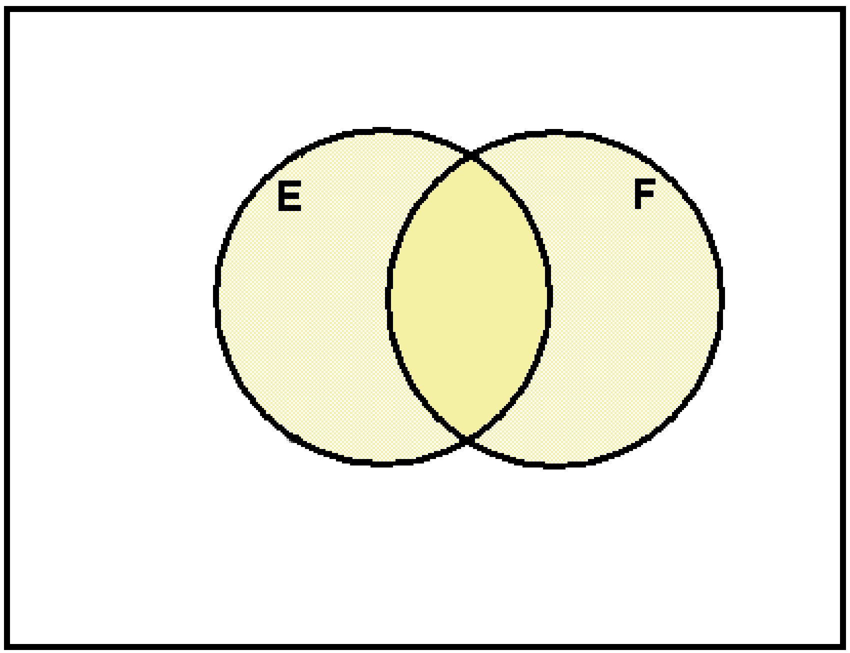
If a random sample of size n is taken from a population of size N, the theoretical probabilities of successive selections calculated on the basis of sampling with replacement and on the basis of sample without replacement differ by insignificant amounts under suitable circumstances.

Typically independence is assumed for the purposes of calculating probabilities when the sample size n is less than 5% of the population size N.

6.6: General Addition Rule for Two Events

For any two events E and F,

$$P(E \cup F) = P(E) + P(F) - P(E \cap F)$$



6.6: General Addition Rule for Two Events

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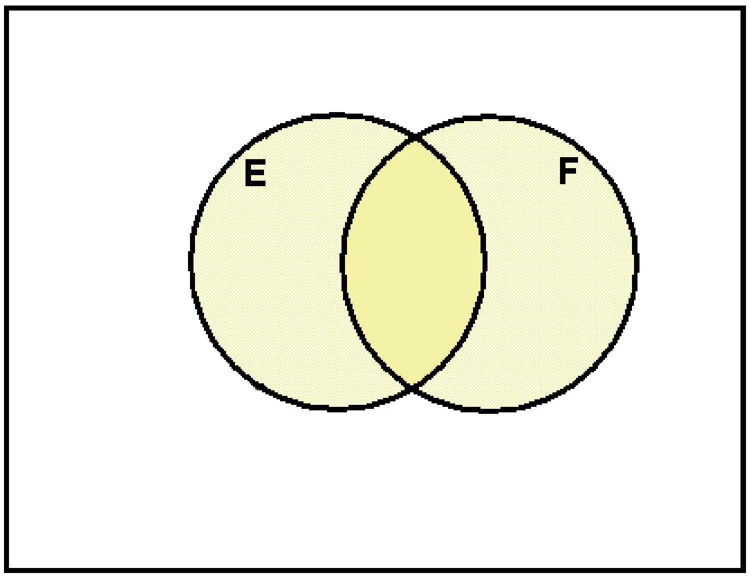
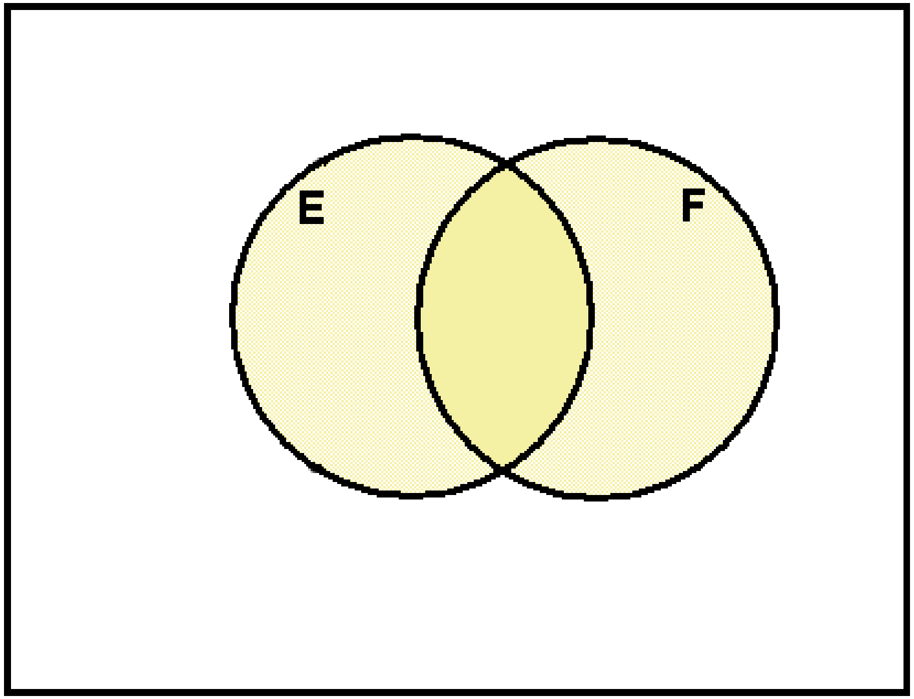
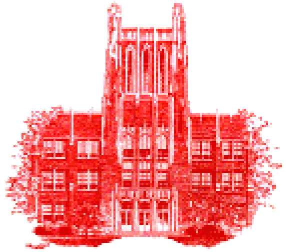


Given: A = drawing a 7 or 8
B = drawing a Club
If you draw one card
What is the $P(A \cup B)$?
What is the $P(A \cap B)$?


E
F



Example

Consider the teacher satisfaction survey

		Job Satisfaction		
		Satisfied	Unsatisfied	Total
L E V E L	College	0.095	0.055	0.150
	High School	0.288	0.220	0.508
	Elementary	0.162	0.180	0.658
	Total	0.545	0.455	1.000

$P(C) = 0.150$, $P(S) = 0.545$ and

$P(C \cap S) = 0.095$, so

$$P(C \cup S) = P(C) + P(S) - P(C \cap S)$$

$$= 0.150 + 0.545 - 0.095$$

$$= 0.600$$

General Multiplication Rule

For any two events E and F,

$$P(E \cap F) = P(E \mid F)P(F)$$

From symmetry we also have

$$P(E \cap F) = P(F \mid E)P(E)$$

Example

18% of all employees in a large company are secretaries and furthermore, 35% of the secretaries are male. If an employee from this company is randomly selected, what is the probability the employee will be a secretary and also male.

Let E stand for the event the employee is male.

Let F stand for the event the employee is a secretary.

The question can be answered by noting that P(F) = 0.18 and P(E|F) = 0.35 so

$$P(E \cap F) = P(E \mid F)P(F) = (0.35)(0.18) = 0.063$$

Bayes Rule

If B_1 and B_2 are disjoint events with

$P(B_1) + P(B_2) = 1$, then for any event E

$$P(B_1|E) = \frac{P(E|B_1)P(B_1)}{P(E|B_1)P(B_1)+P(E|B_2)P(B_2)}$$

More generally, if $B_1, B_2, \ldots, B_k$ are disjoint events with

$P(B_1) + P(B_2) + \ldots P(B_k) = 1$, then for any event E

$$P(B_i \mid E) = \frac{P(E|B_i)P(B_i)}{P(E \mid B_1)P(B_1) + P(E \mid B_2)P(B_2) + \cdots + P(E \mid B_k)P(B_k)}$$

Use when given $P(E|B_1)$ & you want $P(B_1|E)$

Example

A company that makes radios, uses three different subcontractors (A, B and C) to supply on switches used in assembling a radio. 50% of the switches come from company A, 35% of the switches come from company B and 15% of the switches come from company C. Furthermore, it is known that 1% of the switches that company A supplies are defective, 2% of the switches that company B supplies are defective and 5% of the switches that company C supplies are defective.

Thus, we know the defective rate given a switch is from a specific company.

Example

If a radio from this company was inspected and failed the inspection because of a defective on switch, what are the probabilities that that switch came from each of the suppliers.

So we want to find the probability it came from a specific company given the switch is defective. This is the opposite of what we were given, so we use Bayes Rule.

Example - continued

Define the events

S_1 = event that the on switch came from subcontractor A

S_2 = event that the on switch came from subcontractor B

S_3 = event that the on switch came from subcontractor C

D = event the on switch was defective

From the problem statement we have

$$P(S_1) = 0.5,\ P(S_2) = 0.35\ ,\ P(S_3) = 0.15$$

$$P(D|S_1) = 0.01,\ P(D|S_2) = 0.02,\ P(D|S_3) = 0.05$$

Example - continued

P(Switch came from supplier A given it was defective) =

$$P(S_1 \mid D) = \frac{P(D|S_1)P(S_1)}{P(D \mid S_1)P(S_1) + P(D \mid S_2)P(S_2) + P(D \mid S_3)P(S_3)}$$

$$= \frac{(.5)(.01)}{(.5)(.01) + (.35)(.02) + (.15)(.05)}$$

$$= \frac{.005}{.005 + .007 + .0075} = \frac{.005}{.0195} = .256$$

Similarly,

$$P(S_2|D) = \frac{(.35)(.02)}{(.5)(.01)+(.35)(.02)+(.15)(.05)}, \quad P(S_2|D) = .359$$

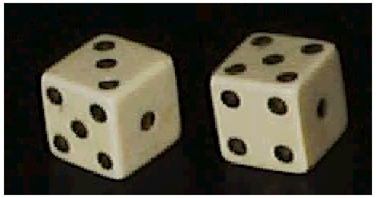
$$P(S_3|D) = \frac{(.15)(.05)}{(.5)(.01)+(.35)(.02)+(.15)(.05)}, \quad P(S_3|D) = .385$$

Example - continued

These calculations show that 25.6% of the defective switches are supplied by subcontractor A, 35.9% of the defective on switches are supplied by subcontractor B and 38.5% of the defective on switches are supplied by subcontractor C.

Even though subcontractor C supplies only a small proportion (15%) of the switches, it supplies a reasonably large proportion of the defective switches (38.9%).

6.7: Estimating Probabilities Empirically & Using Simulation



Estimating Probability Empirically

Common to use observed long term proportions to estimate probabilities empirically.

- Observe large # of chance outcomes in a controlled environment
- Using your knowledge of long run relative frequencies & the law of large numbers estimate the probability of the observed event.

Example

Men & women frequently express intimacy via touch. Holding hands is an example. Some researchers say not only does this act indicate intimacy, but may also indicate status differences.

Research indicates that the males predominately assume the overhand status, women the underhand… The authors of “Men & Women Holding Hands: Whose Hand is Uppermost” believe height may be the more reasonable explanation.

Example…

	Number of hand holding couples		
	Gender of Person with uppermost hand		
	Men	Women	Total
Man taller	2149	299	2448
Equal Ht	780	246	1026
Woman taller	241	205	446
Total	3170	750	3920

Example

Assuming that the reported hand holding couples are representative of the population of hand holding couples, we can estimate various probabilities, i.e

P(man's uppermost) = 3170/3920 = 0.809

P(woman's uppermost) = 750/3920 = 0.191

P(man taller uppermost) = 2149/2448 = 0.878 conditional

P(woman taller uppermost) = 205/446 = 0.460 conditional

So men still have the upperhand

Note: P(man taller uppermost) ≠ P(man's uppermost), therefore NOT independent events

Estimating Probability Simulation

When impractical to measure empirically or can't measure analytically..

- Design method that uses a random mechanism to represent an observation.
- Generate an obs using your method & determine if the outcome of interest has occurred. Repeat a large # of times.
- Calculate the estimated probability by dividing the # of obs for which the outcome of interest occurred by the total # of obs

Example

Lets do a simulation for the multiple choice portion of a 20 question test with 5 choices for each question (only 1 being correct). Using a random # table we randomly pick a place to start & get the following #'s (note take 20 #'s at a time since there are 20 questions).

Since there are 5 choices & 10 possibilities for each of the 20 digits we need 2 #'s to represent success for each of the 20 digits (e.g. Lets let 0 & 1 be success)

Test #1: 9 4 6 0 6 9 7 8 8 2 5 2 9 6 0 1 4 6 0 5   4 correct

Test #2: 6 6 9 5 7 4 4 6 3 2 0 6 0 8 9 1 3 6 1 8   4 correct

Test #3: 0 7 1 7 7 2 9 5 4 8 6 2 7 5 1 0 4 3 0 7   5 correct

etc…